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SURFACES WITH ISOTHERMAL REPRESENTATION OF THEIR LINES OF CURVATURE AS ENVELOPES OF ROLLING.

BY LUTHER PFAHLER EISENHART.

In a former memoir* we have established the existence of transformations of a surface Σ with isothermal representation of its lines of curvature into surfaces Σ_1 of the same kind, such that Σ and a transform Σ_1 envelop a two-parameter family of spheres and the lines of curvature correspond on Σ and Σ_1. Envelopes of spheres possessing the latter property were considered by Ribaucour and are called *transformations of Ribaucour.* It is a property of these transformations that on the locus of the centers of the spheres the curves corresponding to the lines of curvature on Σ and Σ_1 form a conjugate system.

In a recent note Bianchi† developed the idea of the envelope of a plane invariably fixed to a surface S_0 as the latter rolls over an applicable surface S,‡ called the *surface of support.* He called Σ an envelope of rolling and showed that given any non-developable surface Σ the problem of finding pairs of applicable surfaces S_0 and S, such that, as S_0 rolls on S, a plane conveniently fixed with respect to S_0 envelops Σ, requires the solution of a partial differential equation of the second order. In a later note§ Bianchi considered the case where the envelope of rolling Σ is a surface with isothermal representation of its lines of curvature, and found that a transformation of the kind referred to above of Σ into a surface Σ_1 with isothermal representation of its lines of curvature, called by him a *transformation* E_m, gives a solution of the problem. In fact, the locus of the centers of the spheres is the surface of support and the corresponding rolling surface S_0 is found by quadratures.

In the present paper we solve the converse problem: *To show that the transformations* E_m *are the only transformations of Ribaucour for which the given surface is an envelope of rolling with the locus of the centers of the spheres the surface of support.* In order to establish this result we prove also that the transformations E_m are the only transformations of Ribaucour for which the correspondence of the spherical representations of the lines of curvature of the two sheets of the envelope of spheres is conformal.*

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* Transactions of the American Mathematical Society, vol. 9 (1908), pp. 149–177.

† Rendiconti della R. Accademia dei Lincei, vol. 23 (1914), pp. 3–12.

‡ For the meaning of the term rolling as here used see Darboux, Leçons sur la théorie générale des surfaces, Fourth Part, Chapter VI.

§ Rendiconti della R. Accademia dei Lincei, vol. 24 (1915), pp. 366–377.

2. **Envelope of Rolling.** Let Σ be a surface referred to its lines of curvature; ρ_1, ρ_2 be the principal radii of normal curvature; X, Y, Z the direction-cosines of the normal to Σ; and

$$ds^2 = Edu^2 + Gdv^2 \tag{1}$$

the linear element of Σ. Let M be a point on Σ and M_0 the corresponding point on S, a surface of support. Bianchi has shown that M_0 is on the normal to Σ at M. Hence if x, y, z and x_0, y_0, z_0 are the cartesian coördinates of M and M_0 respectively, we have

$$x_0 = x - RX, \qquad y_0 = y - RY, \qquad z_0 = z - RZ, \tag{2}$$

where R is a function of u and v to be determined. From (2) it is readily found that the linear element of S is

$$ds_0^2 = L_1^2du^2 + L_2^2dv^2 + dR^2,$$

where we have put

$$L_1 = \sqrt{E}\left(1 + \frac{R}{\rho_1}\right), \qquad L_2 = \sqrt{G}\left(1 + \frac{R}{\rho_2}\right). \tag{3}$$

Evidently Σ is one of the sheets of the envelope of the spheres of radius R and center M_0. It is a known property of envelopes of spheres that as the locus of the centers is deformed the two points of contact of each sphere with the envelope retain invariable positions.† Hence when S is applied to S_0, the planes tangent to Σ assume one and the same position in space. If this plane be taken for the xy-plane, the coördinates of S_0 are x, y, R, and consequently its linear element is

$$dx^2 + dy^2 + dR^2.$$

Accordingly as S and S_0 are applicable, we must have

$$L_1^2du^2 + L_2^2dv^2 = dx^2 + dy^2. \tag{4}$$

Since the curvature of the quadratic form on the right is zero, the same must be true of the left hand member. Expressing this condition,‡ we obtain the equation of Bianchi

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* In the Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915), pp. 153–176 we established transformations Ω of a surface S into surfaces S_1 such that the common conjugate systems on S and on S_1 have equal tangential invariants and the lines joining corresponding points on S and S_1 form a congruence whose developables meet S and S_1 in these conjugate systems. It was shown that the transformations E_m are the only transformations Ω for which the surfaces S and S_1 envelop a two-parameter family of spheres.

† Bianchi, Lezioni di Geometria Differenziale, vol. 2, p. 88.

‡ Cf. the author's Differential Geometry, p. 155.

$$\frac{\partial}{\partial u}\left(\frac{\sqrt{G}}{\rho_2}\frac{1}{L_1}\frac{\partial R}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{\sqrt{E}}{\rho_1}\frac{1}{L_2}\frac{\partial R}{\partial v}\right)-\frac{\sqrt{EG}}{\rho_1\rho_2}=0. \tag{5}$$

Conversely, when R satisfies this condition, the reduction of the left hand member of (4) to the form of the right hand member requires at most one quadrature, after which the coördinates of S_0 are known.

3. **Transformations of Ribaucour.** Darboux* has shown that if λ and μ are such that

$$\frac{\partial \lambda}{\partial u}+\rho_1\frac{\partial \mu}{\partial u}=0, \qquad \frac{\partial \lambda}{\partial v}+\rho_2\frac{\partial \mu}{\partial v}=0,$$

the envelope of the spheres of radius λ/μ and center

$$\xi = x-\frac{\lambda}{\mu}X, \qquad \eta = y-\frac{\lambda}{\mu}Y, \qquad \zeta = z-\frac{\lambda}{\mu}Z, \tag{6}$$

gives the most general transformation of Ribaucour of Σ. We note that the preceding equations are of the same form as the Rodriques equations for Σ, namely

$$\frac{\partial x}{\partial u}+\rho_1\frac{\partial X}{\partial u}=0, \qquad \frac{\partial x}{\partial v}+\rho_2\frac{\partial X}{\partial v}=0.$$

Let X_1, Y_1, Z_1 and X_2, Y_2, Z_2 denote the direction-cosines of the tangents to the curves v = const. and u = const. respectively on Σ. The cartesian coördinates x_1, y_1, z_1 of a transform Σ_1 of Σ are given by equations of the form

$$x_1 = x-\frac{1}{\sigma m}(\mu X+\alpha X_1+\beta X_2), \tag{7}$$

where m is a constant and the other functions satisfy the following completely integrable system of differential equations:†

$$\begin{gathered}
\frac{\partial \lambda}{\partial u}=\sqrt{E}\alpha, \qquad \frac{\partial \lambda}{\partial v}=\sqrt{G}\beta, \\
\frac{\partial \mu}{\partial u}=-\frac{\sqrt{E}}{\rho_1}\alpha, \qquad \frac{\partial \mu}{\partial v}=-\frac{\sqrt{G}}{\rho_2}\beta, \\
\frac{\partial \alpha}{\partial u}=-\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}\beta+\mu\frac{\sqrt{E}}{\rho_1}+m\sigma(k+\sqrt{E}), \\
\frac{\partial \alpha}{\partial v}=\frac{\beta}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}, \qquad \frac{\partial \beta}{\partial u}=\frac{\alpha}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v},
\end{gathered} \tag{8}$$

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* Leçons, vol. 2, p. 383.

† Cf. Eisenhart, Annali di Matematica, vol. 22 (1914), pp. 194–197.

$$(8)\qquad \frac{\partial \beta}{\partial v} = -\frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}\alpha + \mu\frac{\sqrt{G}}{\rho_2} + m\sigma(l + \sqrt{G}),$$

$$\frac{\partial \log \sigma}{\partial u} = k\frac{\alpha}{\lambda}, \qquad \frac{\partial \log \sigma}{\partial v} = l\frac{\beta}{\lambda},$$

$$\frac{\partial k}{\partial v} = l\left[\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v} - \frac{\beta}{\lambda}(k + \sqrt{E})\right],$$

$$\frac{\partial l}{\partial u} = k\left[\frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u} - \frac{\alpha}{\lambda}(l + \sqrt{G})\right],$$

and the quadratic relation

$$(9)\qquad \alpha^2 + \beta^2 + \mu^2 = 2m\lambda\sigma.$$

Conversely every set of functions satisfying equations (8) and (9) determines a transformation of Ribaucour of Σ.

The functions k^2 and l^2 are equal respectively to E_1 and G_1, the first fundamental coefficients of Σ_1. The direction-cosines X_1', Y_1', Z_1'; X_2', Y_2', Z_2' of the tangents to the parametric curves on Σ_1 are given by

$$\frac{\partial x_1}{\partial u} = \sqrt{E_1}X_1', \qquad \frac{\partial x_1}{\partial v} = \sqrt{G_1}X_2',$$

and similar equations in y_1 and z_1. It is necessary to distinguish between two cases, namely

$$(a)\qquad k = -\sqrt{E_1}, \qquad l = \sqrt{G_1},$$

$$(b)\qquad k = \sqrt{E_1}, \qquad l = \sqrt{G_1}.$$

In order that the mutual orientation of the trihedral for Σ_1 shall be the same as for Σ, the direction-cosines of the normal to Σ_1, namely X', Y', Z', are given by equations of the form

$$x_1 - x + \frac{\lambda}{\mu}(X \mp X') = 0,$$

where the upper and lower signs before X' correspond to cases (a) and (b) respectively.

Expressing the condition that the equations of Rodriques for Σ_1 are of the same form as for Σ, we are led to the respective pairs of equations, for the determination of ρ_1' and ρ_2',

$$(10)\qquad \begin{aligned} &\frac{\partial \log \sigma}{\partial u}\left(\pm\frac{1}{\rho_1'} + \frac{\mu}{\lambda}\right) - \frac{\partial}{\partial u}\left(\frac{\mu}{\lambda}\right) = 0, \\ &\frac{\partial \log \sigma}{\partial v}\left(\pm\frac{1}{\rho_2'} + \frac{\mu}{\lambda}\right) - \frac{\partial}{\partial v}\left(\frac{\mu}{\lambda}\right) = 0. \end{aligned}$$

We consider now case (*a*). The corresponding equations (10) are reducible by means of (8) to

$$
\begin{aligned}
&\sqrt{E_1}\left(\frac{1}{\rho_1'}+\frac{\mu}{\lambda}\right)=\sqrt{E}\left(\frac{1}{\rho_1}+\frac{\mu}{\lambda}\right),\\
(11)\qquad&\\
&\sqrt{G_1}\left(\frac{1}{\rho_2'}+\frac{\mu}{\lambda}\right)=-\sqrt{G}\left(\frac{1}{\rho_2}+\frac{\mu}{\lambda}\right).
\end{aligned}
$$

4. **Solution of the problem.** In accordance with the problem as stated we consider now the case where equations (2) and (6) define the same surface. If we put

$$R=\lambda/\mu,$$

it follows from (8) that

$$\frac{\partial R}{\partial u}=\frac{\alpha}{\mu}L_1,\qquad \frac{\partial R}{\partial v}=\frac{\beta}{\mu}L_2.$$

Substituting in equation (5), we reduce the resulting equation by means of (8) and (9) to

$$\frac{\sqrt{G}}{\rho_2}(-\sqrt{E_1}+\sqrt{E})+\frac{\sqrt{E}}{\rho_1}(\sqrt{G_1}+\sqrt{G})+2\frac{\sqrt{EG}}{\rho_1\rho_2}\frac{\lambda}{\mu}=0,$$

which because of (11) is equivalent to

$$(12)\qquad \frac{\sqrt{G}}{\rho_2}\frac{\sqrt{E_1}}{\rho_1'}=\frac{\sqrt{E}}{\rho_1}\frac{\sqrt{G_1}}{\rho_2'}.$$

By similar processes we find for case (*b*) the equation

$$(12')\qquad \frac{\sqrt{G}}{\rho_2}\frac{\sqrt{E_1}}{\rho_1'}+\frac{\sqrt{E}}{\rho_1}\frac{\sqrt{G_1}}{\rho_2'}=0.$$

If the linear elements of the spherical representations of Σ and Σ_1 be written

$$ds'^2=e\,du^2+g\,dv^2,\qquad ds_1'^2=e_1du^2+g_1dv^2,$$

then*

$$\sqrt{e}=\frac{\sqrt{E}}{\rho_1},\qquad \sqrt{g}=\frac{\sqrt{G}}{\rho_2},\qquad \sqrt{e_1}=\frac{\sqrt{E_1}}{\rho_1'},\qquad \sqrt{g_1}=\frac{\sqrt{G_1}}{\rho_2'}.$$

Also we have†

$$(13)\qquad \frac{\partial\sqrt{e}}{\partial v}=\frac{1}{\rho_2}\frac{\partial\sqrt{E}}{\partial v},\qquad \frac{\partial\sqrt{g}}{\partial u}=\frac{1}{\rho_1}\frac{\partial\sqrt{G}}{\partial u}.$$

Consequently equations (12) and (12′) give

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* Cf. the author's Differential Geometry, p. 200.

† Ibid., p. 157.

$$\frac{e}{e_1} = \frac{g}{g_1}, \tag{14}$$

from which it follows that

A necessary and sufficient condition that the locus of the centers of the spheres of a transformation of Ribaucour of a surface Σ *be the surface of support with* Σ *the corresponding envelope of rolling is that the correspondence between the spherical representations of the lines of curvature of the two sheets of the envelope of spheres be conformal.*

5. **Transformations** E_m. Now we shall show that transformations E_m are characterized by the property mentioned in the above theorem.

We replace equation (12) by

$$\frac{\sqrt{E_1}}{\rho_1'} = \rho\frac{\sqrt{E}}{\rho_1}, \qquad \frac{\sqrt{G_1}}{\rho_2'} = \rho\frac{\sqrt{G}}{\rho_2},$$

where the factor of proportionality ρ is to be determined. Now equations (11) may be replaced by

$$\sqrt{E_1} - \sqrt{E} = \sqrt{e}\frac{\lambda}{\mu}(1 - \rho), \qquad \sqrt{G_1}\cdot + \sqrt{G} = -\sqrt{g}\frac{\lambda}{\mu}(1 + \rho). \tag{15}$$

If these values of $\sqrt{E_1}$ and $\sqrt{G_1}$ be substituted in the last two of equations (8), we find that ρ must satisfy the equations

$$\begin{aligned} \frac{\partial}{\partial v}\log\rho &= -\frac{\partial}{\partial v}\log e - \frac{\beta}{\mu}(1 - \rho)g, \\ \frac{\partial}{\partial u}\log\rho &= -\frac{\partial}{\partial u}\log g - \frac{\alpha}{\mu}(1 + \rho)e. \end{aligned} \tag{16}$$

Expressing the condition of integrability of these equations, we get

$$\frac{\partial^2}{\partial u \partial v}\log\frac{e}{g} = 0.$$

Hence

$$\frac{e}{g} = \frac{U}{V},$$

where U and V are functions of u and v respectively, that is, the spherical representation of the lines of curvature of Σ is isothermal; and in consequence of (14) the same is true of Σ_1.

When similar processes are applied to (12′), we get the same equations (16).

By a suitable choice of the parameters we can reduce U and V to unity and then we introduce functions θ and φ by the equations

$$\sqrt{e} = \sqrt{g} = e^{-\theta}, \qquad \rho = e^{2\theta}\mu/\varphi, \tag{17}$$

where the e's of the right hand members are the Naperian base. Now equations (16) reduce to

$$\frac{\partial\varphi}{\partial u} = \alpha e^{\theta}, \qquad \frac{\partial\varphi}{\partial v} = -\beta e^{\theta}, \tag{18}$$

which are readily found to be consistent.

From (15) and (17) we have

$$\sqrt{E_1} = \sqrt{E} + \frac{\lambda}{\mu}\left(e^{-\theta} - e^{\theta}\frac{\mu}{\varphi}\right), \qquad \sqrt{G_1} = -\sqrt{G} - \frac{\lambda}{\mu}\left(e^{-\theta} + e^{\theta}\frac{\mu}{\varphi}\right). \tag{19}$$

Substituting these values in the expressions (8) for $\partial \log \sigma/\partial u$ and $\partial \log \sigma/\partial v$, we have

$$\frac{\partial \log \sigma}{\partial u} = -\frac{\partial \log \lambda}{\partial u} + \frac{\partial \log \mu\varphi}{\partial u}, \qquad \frac{\partial \log \sigma}{\partial v} = -\frac{\partial \log \lambda}{\partial v} + \frac{\partial \log \mu\varphi}{\partial v}.$$

Hence to within a constant factor we have by integration

$$\sigma = \mu\varphi/\lambda. \tag{20}$$

This constant will be taken equal to unity, as the constant m appears in (8) and (9) as a multiplier of σ.

Now the first eight of equations (8) and (18) are reducible to

$$\begin{gathered}
\frac{\partial\lambda}{\partial u} = \sqrt{E}\alpha, \qquad \frac{\partial\lambda}{\partial v} = \sqrt{G}\beta, \\
\frac{\partial\mu}{\partial u} = -e^{-\theta}\alpha, \qquad \frac{\partial\mu}{\partial v} = -e^{-\theta}\beta, \\
\frac{\partial\alpha}{\partial u} = \frac{\partial\theta}{\partial v}\beta + e^{-\theta}\mu - m(e^{-\theta}\varphi - e^{\theta}\mu), \\
\frac{\partial\alpha}{\partial v} = -\frac{\partial\theta}{\partial u}\beta, \qquad \frac{\partial\beta}{\partial u} = -\frac{\partial\theta}{\partial v}\alpha, \\
\frac{\partial\beta}{\partial v} = \frac{\partial\theta}{\partial u}\alpha + e^{-\theta}\mu - m(e^{-\theta}\varphi + e^{\theta}\mu), \\
\frac{\partial\varphi}{\partial u} = \alpha e^{\theta}, \qquad \frac{\partial\varphi}{\partial v} = -\beta e^{\theta}.
\end{gathered} \tag{21}$$

These equations are obtained in case (b) also, and the only change necessary is that of the sign of $\sqrt{E_1}$ in (19).

Equations (21) are equivalent to the equations of a transformation E_m derived by Bianchi.* Hence we have

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* L. c., p. 371.

A necessary and sufficient condition that the correspondence between the spherical representations of the lines of curvature of two surfaces in the relation of a transformation of Ribaucour be conformal is that the spherical representations be isothermal, in which case the transformation is an E_m.

Combining these two theorems, we have the desired result:

Transformations E_m *are the only transformations of Ribaucour for which the given surface is an envelope of rolling with the locus of the centers of the spheres for the surface of support.*

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